OPERATORS WITH DENSE RANGE AND THE INF-SUP CONDITIONS

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Let V and Q be two Hilbert spaces and $\Lambda : V \to Q$ a bounded linear operator with a dense range, i.e.

$$\overline{\operatorname{Ran}\Lambda} = Q.$$

The following standard results on Hilbert spaces will be used:

Lemma 1. *Let $T : V \to Q$ be a bounded linear operator and let $T^* : Q \to V$ be its adjoint, defined by*

$$(T^*q, v)_V := (q, Tv)_Q \quad \forall q \in Q,\ v \in V.$$

Then

(1) $\operatorname{Ker} T = (\operatorname{Ran} T^*)^\perp$,
(2) $(\operatorname{Ker} T)^\perp = \overline{\operatorname{Ran} T^*}$

Lemma 2. *A bounded linear operator $T : V \to Q$ has a closed range if and only if its adjoint $T^* : Q \to V$ has a closed range.*

Also, note that by the Riesz Representation theorem, we can represent the norm of an arbitrary element $u \in V$ as the norm of its corresponding dual:

$$||u||_V = \sup_{v \in V \setminus \{0\}} \frac{(u, v)_V}{||v||_V}. \tag{1}$$

1. Making Λ into a closed operator

Define the new norm $|||\cdot|||_Q$ on Q as follows:

$$|||q|||_Q := ||\Lambda^* q||_V.$$

Note that $|||\cdot|||_Q$ is indeed a norm because, due to Lemma 1 and the fact that the range of Λ is dense in Q, we have that

$$\operatorname{Ker}\Lambda^* = (\operatorname{Ran}\Lambda)^\perp = Q^\perp = \{0\}.$$

Moreover, Λ^* is a bounded operator, and therefore $|||q|||_Q \lesssim ||q||_Q$. As a result, every Cauchy sequence in $|||\cdot|||_Q$ is Cauchy in $||q||_Q$, and the space Q equipped with $|||\cdot|||_Q$ is a Hilbert space.

Now, we define the Hilbert space equipped with the $|||\cdot|||_Q$:

$$\tilde{Q} := \overline{\operatorname{Ran}\Lambda},$$

where the closure is taken with the norm $|||\cdot|||_Q$. We also define $\tilde{\Lambda} : V \to \tilde{Q}$ as Λ but mapped into Q with the new norm. Then, by the definition of the new norm,

$$||\tilde{\Lambda}^* q||_V = |||q|||_Q$$

and therefore $\tilde{\Lambda}^*$ is an isometry into $\tilde{Q}$; in particular, an isometry between Hilbert spaces has a closed range. Indeed, for $q \in \tilde{Q}$, there is a sequence $q_n \in \operatorname{Ran} \Lambda$ such that $q_n \to q$ in $\tilde{Q}$. The sequence q_n is Cauchy and for each n there is a $v_n \in V$ such that $\Lambda^* q_n = v_n$. Then, since $||v_n - v_m||_V = ||\Lambda^*(q_n - q_m)||_V = |||q_n - q_m|||_Q$, the sequence v_n is also Cauchy in V and converges. By continuity, $q = \lim_n \Lambda^* v_n$ and $q \in \operatorname{Ran} \Lambda^*$.

Using Lemma 2, we establish that $\tilde{\Lambda}$ **has a closed range**. We can write the norm $|||\cdot|||_Q$ as Schöberl does using (1):

$$|||q|||_Q = \sup_{v \in V \setminus \{0\}} \frac{(\Lambda^* q, v)_V}{||v||_V} = \sup_{v \in V \setminus \{0\}} \frac{(q, \Lambda v)_V}{||v||_V}.$$

2. Closed range and inf-sup conditions

Assume that $\Lambda^* : Q \to V$ is a bounded linear operator with a closed range (I remove the tildes from before for simplicity). Since V is a Hilbert space, then $\operatorname{Ran} \Lambda^*$ is a Hilbert space equipped with the inner product from V. Then,

$$\Lambda^* : Q \to \operatorname{Ran} \Lambda$$

is a **surjective** map between Hilbert spaces.

Consider the quotient space $Q / \operatorname{Ker} \Lambda^*$. This is defined as the "set of sets"

$$q + \operatorname{Ker} \Lambda^* = \{p \in Q : \exists r \in \operatorname{Ker} \Lambda^* \quad \text{s.t.} \quad p = q + r\}$$

with norm

$$||q + \operatorname{Ker} \Lambda^*||_{Q/\operatorname{Ker} \Lambda^*} = \inf_{r \in \operatorname{Ker} \Lambda^*} ||q + v||_Q.$$

For simplicity, I'll write q instead of $q + \operatorname{Ker} \Lambda^*$ for elements in the quotient space. We can then consider the quotient operator

$$\hat{\Lambda}^* : \frac{Q}{\operatorname{Ker} \Lambda^*} \to \operatorname{Ran} \Lambda^*$$

defined by

$$\hat{\Lambda}^*(q + \operatorname{Ker} \Lambda^*) := \Lambda^* q.$$

This operator is automatically **injective**. Moreover, it is a bijective bounded linear map between Hilbert spaces. By the inverse function theorem, such a map has a bounded inverse. In particular, this implies that there is a $C > 0$ such that

$$C||q||_{Q/\operatorname{Ker} \Lambda^*} \leq ||\hat{\Lambda}^* q||_V \quad \forall q \in Q/\operatorname{Ker} \Lambda^*.$$

Since we are working on Hilbert spaces we may use (1) to rewrite this condition as the inf-sup condition:

$$C||q||_{Q/\operatorname{Ker} \Lambda^*} \leq \sup_{v \in V \setminus \{0\}} \frac{(\hat{\Lambda}^* q, v)_V}{||v||_V} = \sup_{v \in V \setminus \{0\}} \frac{(q, \Lambda v)_V}{||v||_V} \quad \forall q \in Q/\operatorname{Ker} \Lambda^*.$$

Remark 3. *Regarding the Stokes equation, the deep result discovered by Olga Ladyzhenskaya and others was that*

$$\operatorname{grad} : L^2(\Omega) \to H^{-1}(\Omega)$$

has a closed range. Its kernel Ker (grad) *is equal to the space of constant functions and its adjoint is the divergence operator. By quotienting* Ker (grad) *out of* $L^2(\Omega)$ *we obtain* $L^2_0(\Omega)$*, which is known to be inf-sup stable together with* $H^1_0(\Omega)$*.*